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APPROXIMATE DETERMINATION OF THE MAGNITUDE OF THE CRITICAL SIZE IN THE PROBLEM OF THE EVOLUTION OF AN IMPACT

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The evolution of a nonlinear dynamic system after the action of a localized effect (impact) is examined. A semiquantitative method is suggested for determining the critical size of the impact upon exceeding which the diffusional mode of decay in the system is replaced by a wave mode of propagation. The use of the proposed technique to evaluate the critical size of an impact in a realistic model of an oceanic ecological system is demonstrated.

Introduction. The wave properties of diffusion-reaction systems were examined for the first time in [1, 2] in regard to the biological problem of the propagation of a gene having an advantage in the struggle for existence and in [3] in connection with problems of the propagation of a flame. In [1, 2] it is shown that for an initial condition of the type of a "step" a running wave is formed asymptotically in such systems. A different situation may arise if the initial condition has the form of a local perturbation, i.e., is a finite function. The statement of this problem is due to A. N. Kolmogorov (see [4]). If turns out that for a certain class of diffusion-reaction systems two qualitatively different regimes of evolution of a system are possible in accordance with the size of the perturbation (which will be also called a damage or impact). When the size of an impact is smaller than a certain critical value, the perturbation is noticeably different from zero only in a limited region and decays with time. If the size of an impact exceeds the critical value, the damage propagates, with the propagation having the character of a running wave. In such a situation, the problem of determining the magnitude of the critical size is of interest in its own right. In the present work a technique for estimating this value on the basis of separate account for the contributions of diffusion and reaction is proposed. It should be noted that a mathematical investigation of this problem was carried out in [5].

Problem of the Evolution of an Impact. Consider an active medium described by the nonlinear diffusion-reaction equation

$$\partial_t u = D\partial^2_{xx} u + (1/\tau) f(u) , \tag{1}$$

where $u(x, t)$ is a function that describes the state of the medium (for example, the concentration of a chemically active material or the population density of specimens); we assume that $0 \le u \le 1$; D is the diffusion coefficient; τ is a parameter with the dimension of time that characterizes the strength of the source. Suppose the initial perturbation of the system is a finite function of the following form (a "rectangular impact"):

$$u(x, 0) = \begin{cases} u_1 , & |x| < \Delta/2 ; \\ 0 , & |x| > \Delta/2 . \end{cases} \tag{2}$$

Equation (1) was examined for the first time in [1, 2] for an initial condition of the type of a "step": $u(x, 0) = 1$, $x > 0$; $u(x, 0) = 0$, $x < 0$, and for a source function with the properties

$$f(0) = f(1) = 0 ; \quad f(u) > 0 , \quad 0 < u < 1 ; \tag{3}$$

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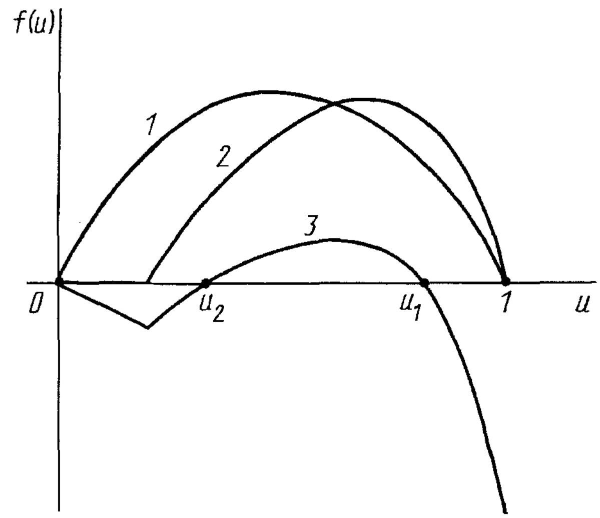


Fig. 1. Form of the source function: curve 1) in accordance with [1, 2]; 2) according to [3]; 3) the same with account for heat transfer (processes of natural death). All the quantities in Figs. 1–3 are dimensionless.

$$f'(0) > 0 ; \quad f'(u) < f'(0) , \quad 0 < u \le 1 \tag{4}$$

(the form of $f(u)$ is depicted qualitatively in Fig. 1, curve 1).

It should be noted that the general properties of the system are determined by the properties of the source. For Eq. (1) with a source of the form (3), (4) an asymptotic solution of the running wave type exists not only for an initial condition of the type of a "step" but also for an arbitrary finite perturbation (that does not vanish identically), in particular, for a "rectangular impact" (2). This holds for any values of the determining parameters D, τ, and Δ.

In many cases the source function does not satisfy condition (4) (for example, the Arrhenius law for the kinetics of chemical reactions, the law of the growth in the number of specimens for Alle-type populations (see [6, 7])). In [3], a source of the following form was suggested (see curve 2, Fig. 1):

$$f_1(u) \equiv 0 , \quad 0 \le u \le a ; \quad f_1(u) > 0 , \quad f < u < 1 ; \quad f_1(1) = 0 . \tag{5}$$

In this case the situation changes substantially. If the width of the impact Δ is rather large in a dynamic system governed by Eq. (1) with source (5), the perturbation grows in the entire region; for small values of Δ the initial perturbation decays with time. A mathematically rigorous investigation of this problem was made in [5].

If we take into account processes of heat transfer with the external medium (for diffusion-reaction systems; correspondingly; processes of natural mortality for aggregations of living organisms), then the source function takes the following form (curve 3, Fig. 1):

$$f(u) = f_1(u) - \lambda u , \tag{6}$$

where λ is a certain constant determined by the properties of the system.

Problem (1) and (2) with a source of type (5) and (6) was investigated in [8, 9] in connection with the problem of the appearance of impact waves in ecological systems. It is shown that when $A \equiv \int_0^{u_1} f(u)du > 0$ (in this case the wave propagates to the side of the region with a smaller value of the stationary solution – the so-called "wave of population"; see [10]) two qualitatively different regimes of the solution can occur in accordance with the relationship between the governing parameters of the problem [4]: the diffusional regime in which the initial perturbation decays with time and the wave regime with the formation of two waves going in different directions (note that when $A < 0$, which corresponds to the "wave of extinction," "collapse" of the impact occurs irrespective of the values of the parameters).

Actually, in terms of the dimensionless variables $t' = t/\tau$ and $x' = x/(D\tau)^{1/2}$ problem (1), (2) takes the form

$$\partial_{t'}u = \partial^2_{x'x'}u + f(u), \quad u(x', 0) = \begin{cases} u_1, & |x'| < \varepsilon/2; \\ 0, & |x'| > \varepsilon/2. \end{cases} \tag{7}$$

where $\varepsilon = \Delta/(D\tau)^{1/2}$ (hereafter the primes on the variables x' and t' will be omitted). Thus, the solution of dimensionless problem (7) is $u = u(x, t; \varepsilon)$, and the global properties of the solution are determined by the value of the dimensionless parameter ε, and the asymptotic formation of the wave corresponds to the region of large values of ε and diffusional decay corresponds to the region of small values of it. Consequently, a critical value ε_{cr} should exist at which bifurcation of one type of solution to the other takes place, with the value of ε_{cr} depending only on the form of the source function $f(x)$. Note that for a specific diffusion-reaction system the values of D and τ are fixed and the value of ε_{cr} unambiguously corresponds to the critical dimension Δ_{cr} of the impact.

The conclusion of the existence of ε_{cr} drawn in [8, 9] was supported by results of numerical experiments. In this case, for a source of type (6) where

$$f_1(u) = \begin{cases} 0, & 0 \le u \le a; \\ 4 f_{max}(u - a)(1 - u)/(1 - a)^2, & a < u \le 1 \end{cases} \tag{8}$$

(for the values of the parameters $f_{max} = 0.1$, $a = 0.2$, and $\lambda = 0.05$) the value 1.9 was obtained. Note that a rather large number of numerical experiments makes it possible in principle to determine ε_{cr} with any required accuracy, but this process is extremely laborious and requires large expenditures of computer time. It is of interest to develop analytical methods of determining ε_{cr}. The present paper suggests one of the possible means of estimating this parameter.

Estimation of the Critical Parameter. The idea behind the technique consists in the following. Equation (1) is nonlinear, and therefore we cannot separate the contributions to $u(x, t)$ from each of the processes: diffusional spreading of the initial perturbation and variations caused by the source. However, we may try to compare the rates of variation of $u(x, t)$ due to each of these processes, since $\partial u/\partial t$ is a sum of the rates of diffusion and action of the source. Such an approach cannot, of course, claim high accuracy.

Let us consider the Cauchy problem for a linear homogeneous diffusion equation

$$\partial_t u = \partial^2_{xx}u, \quad u(x, 0) = \begin{cases} u_1, & |x| < \varepsilon/2; \\ 0, & |x| > \varepsilon/2. \end{cases} \tag{9}$$

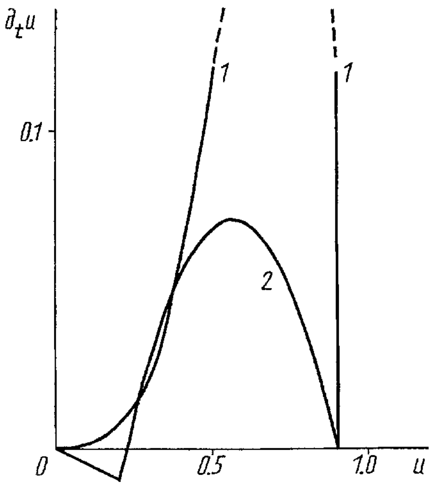
The solution of this problem is well known:

$$u^{dif}(x, t) = (u_1/2)\left\{\Phi\left(\frac{x + 0.5\varepsilon}{(4t)^{1/2}}\right) - \Phi\left(\frac{x - 0.5\varepsilon}{(4t)^{1/2}}\right)\right\}, \tag{10}$$

where $\Phi(\xi)$ is the probability integral. Correspondingly, the rate of change of the diffusion solution is

$$\partial_t u^{dif}(x, t) = u_1 (16\pi t^3)^{-1/2}\left\{(x - 0.5\varepsilon)\exp\left[-\frac{(x - 0.5\varepsilon)^2}{4t}\right] - \right.$$
$$\left. - (x + 0.5\varepsilon)\exp\left[-\frac{(x + 0.5\varepsilon)^2}{4t}\right]\right\}. \tag{11}$$

First of all, we are interested in the behavior of the solution at the center of the impact, i.e., at $x = 0$. In fact, it can be easily seen that $a(0, t) = \max_{(x)}u(x, t)$ (for the initial condition from Eq. (9)) and the fulfillment of



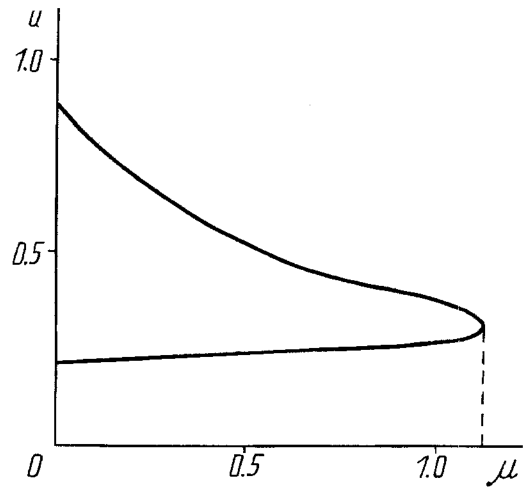


Fig. 2. Curves of the rate of change of the solution vs the value of the solution at the impact center: curve 1, rate of decrease due to diffusional spreading; 2, rate of growth due to the action of a source.

Fig. 3. Positions of the points of intersection of the curves for the rates of growth and decay of a solution as a function of the controlling parameter μ.

the condition $u(0, t_0) < u_2$ at a certain value t_0 is a criterion for the development of the diffusional regime of the process.

The rate of decrease in the solution at $x = 0$ due to diffusion is

$$\partial_t u^{\text{dif}}(0, t) = -u_1 \varepsilon (16\pi t^3)^{-1/2} \exp(-\varepsilon^2/16t). \tag{12}$$

Let us try to express the diffusional velocity (12) in terms of the solution at the center of the impact. Let $\Phi^{-1}(z)$ be a function reciprocal to the probability integral Φ; then from

$$u^{\text{dif}}(0, t) = u_1 \Phi(\varepsilon/(16t)^{1/2}) \tag{13}$$

we have

$$\varepsilon/(16t)^{1/2} = \Phi^{-1}(u^{\text{dif}}(0, t)/u_1). \tag{14}$$

Substituting Eq. (14) into Eq. (12), we obtain

$$\partial_t u^{\text{dif}}(u) = -\mu\, [\Phi^{-1}(u/u_1)]^3 \exp\left\{-[\Phi^{-1}(u/u_1)]^2\right\}, \tag{15}$$

where $\mu = 16\pi^{-1/2} u_1 \varepsilon^{-2}$. We can easily see that the special function

$$g(z) = [\Phi^{-1}(z)]^3 \exp\left\{-[\Phi^{-1}(z)]^2\right\} \tag{16}$$

exhibits the following properties: a) it is positive over the interval (0, 1); b) it has a single maximum at $z = z_0 = \Phi(3/2)^{1/2} \approx 0.92$ (in this case $g(z) \approx 0.4$); c) $g'(z) \to \infty$ as $z \to 1 (z < 1)$. The graph of the function $g(z = u/u_1)$ is given in Fig. 2 (curve 1). Note that $g(u/u_1)$ has the meaning of the rate of diffusional decrease of the solution at the center of the impact (accurate to the constant μ).

Figure 2 presents the graph (curve 2) of the rate of growth of the solution due to the action of the source

$$\partial_t u^{\text{source}}(u) = f(u) \tag{17}$$

(where $f(u)$ is prescribed in the form of Eq. (8)). In the approximation considered – "splitting" of the processes of local growth due to the source and diffusion – the relative position of the curves $f(u)$ and $\mu g(u/u_1)$ determines the global properties of the solution of problem (7). If (in accordance with the value of μ) curve 1 lies above curve 2, diffusional spreading predominates. If the curves intersect, the rate of growth due to the source prevails, and a wave regime develops in the system. The dependence of the coordinates of the points of intersection of the curves on the value of μ is presented in Fig. 3; the limiting value μ_{cr} found amounts to ≈ 1.12, whence $\varepsilon_{cr} \approx 2.7$ (with the "true" 1.9 value determined in [8, 9] by numerical simulation).

Application of the Method in the Model of an Ecological System. Let us try to apply the above-described method for estimating the critical size of an impact in a realistic model of an oceanic ecological system proposed in [11, 12]. Investigation of the evolution of an impact by the methods of computer simulation was performed in [8, 9]. Let us consider the following system of equations that describes the functioning of an ecological system after the introduction of an impact – certain living organisms feeding on phytoplankton:

$$\partial_t u = D\,\partial_{xx}^2 u + \alpha\, u\, P_p - \lambda u\,, \tag{18}$$

$$\partial_t u_p = D\,\partial_{xx}^2 u_p + (1 - \beta\, u)\, P_p - E_p - M_p\,, \tag{19}$$

$$\partial_t u_i = D\,\partial_{xx}^2 u_i + P_i - E_i - M_i\,, \quad i = 1\,,\ 2\,,\ \ldots,\ 8\,. \tag{20}$$

Here the function $u(x, t)$ is the concentration (population density) of the organisms introduced; $u_p(x, t)$ is the concentration of the phytoplankton; $u_i(x, t)$ is the concentration of the remaining components involved in the ecological system (zooplankton, bacteria, protozoa, etc.); P are the production (the rate of reproduction) of the corresponding components; E are the eating-away; M are the dying-out; D is the coefficient of turbulent diffusion; α, β, λ are constants.

The production of phytoplankton depends on its concentration and on the concentration of biogenic elements dissolved in water $u_8(x, t)$

$$P_p = C u_p u_8/(u_8 + K)\,, \tag{21}$$

where K is the Michaelis-Menton constant (see [13]); C is a proportionality factor (generally depending on the water temperature). In turn, the concentration of the biogenic elements is

$$u_8 = a + bu \tag{22}$$

(a and b are coefficients that depend on the concentrations of the remaining components of the system).

With the use of Eqs. (21) and (22), Eq. (18) yields

$$\partial_t u = D\,\partial_{xx}^2 u + \alpha_1 u u_p\,(a + bu)/[(a + K) + bu] - \lambda u\,. \tag{23}$$

In an actual ecological system the concentration of phytoplankton depends in a certain way on the concentration of its predators (in the present case, the organisms introduced, the development of whose population is of interest to us). The main assumption required to split the system of equations (18)–(20) is the assignment of a specific form of this dependence. In this regard, the following condition was considered in [14]:

$$u + u_p = M \approx \text{const}\,, \tag{24}$$

whence

$$u_p = M - u\,. \tag{25}$$

Substituting Eq. (25) into Eq. (23) and assuming additionally that $a \approx$ const, $b \approx$ const, we have (relabeling the constants)

$$\partial_t u = D\,\partial^2_{xx} u + [(1/\tau_1)\,u\,\varphi(u)\,(1-u) - (1/\tau_2)\,u]\,. \tag{26}$$

Here τ_1 has the meaning of the characteristic time of replacement of generations, τ_2 of the characteristic lifetime of one specimen. Equation (26) obtained in this way is a reduction of the initial "full" system (18)-(20) when the effect of the remaining components of the ecological system is taken into account effectively via a special form of the source function. Assuming that τ_1 = 1 day and τ_2 = 50 days (which correspond to primitive living organisms) and using the estimation technique given above, we have $\Delta_{cr} \approx 160$ km. Note that Eq. (26) was derived on the basis of rather strong assumptions, and therefore we may speak only about the estimate of the order of magnitude of Δ_{cr}.

By virtue of assumption (25), $u_p = 0$ when $u = M$. This seems to be not entirely realistic. We will consider Eq. (25) to be the first term in an expansion of the function $u_p(u)$ in powers of u and assume that

$$u_p = M \exp(-u/M)\,. \tag{27}$$

Substituting Eq. (27) into Eq. (23) (the resulting equation has the same form as Eq. (26) with a certain quantitative difference in the behavior of the source), we obtain $\Delta_{cr} \approx 45$ km for the given values of τ_1 and τ_2.

Thus, the obtained value of about 100 km for the critical size of an impact (for the system of equations (18)–(20)) is quite stable to small "perturbations" of the scheme considered. Note that slight variations in the parameters τ_1 and τ_2 also do not have a significant effect on Δ_{cr}.

Conclusion. A semiquantitative approach has been proposed for determining the critical size of an impact (damage) in a nonlinear dynamic system of diffusion-reaction type. When the critical size is exceeded, the damage starts to propagate throughout the system in the form of a travelling wave.

The possibility of using this approach in practice has been demonstrated on the example of a realistic model of an oceanic ecological system. The value ~100 km obtained for the critical size of the "afflicted" area of an ocean is quite stable to variations in the model parameters.

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NOTATION

x, coordinate; t, time; D, diffusion coefficient; Δ, width of an impact; u_1, impact intensity; τ, a, λ, A, μ, parameters of a diffusion-reaction system determined in situ; Φ, probability integral; g, special function determined in situ; P, E, M, production, eating-away, and dying-out of the components of an ecological system; α, β, C, K, a, b, M, τ_1, τ_2, parameters of an ecological system determined in situ.

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